EFFECT OF LARMOR PRECESSION OF CHARGED PARTICLES ON NONSTATIONARY TEMPERATURE FIELD IN A FLAT CHANNEL

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Under the assumption of a small magnetic Reynolds number an exact solution was obtained in [1] for the problem of nonstationary plane-parallel flow of an ionized medium in a trnsverse magnetic field taking into account the effect of Larmor precession of electrons and ions. On the basis of results from [1] for the case of constant pressure drop the temperature distribution is found in this paper over the cross section of a flat channel under conditions of constant channel wall temperature. All assumptions adopted in the paper [1] are retained.

1. By virtue of assumptions made in [1] with regard to incompressibility of the medium, with regard to constancy of its degree of ionization and with regard to smallness of the magnetic Reynolds number, energy equations can be written in the form [2]

$$C_v^\circ (1 + Zs)\, \rho \left(\frac{\partial T}{\partial t} + \mathbf{u}\nabla T\right) = -\operatorname{div} \mathbf{q} - \pi^{rm} \frac{\partial u^r}{\partial x_m} + \mathbf{j}\,(\mathbf{E}_0 + \mathbf{u} \times \mathbf{B}_0) \qquad (1.1)$$

where T is the temperature, C_v° is the heat capacity per unit mass of the neutral gas, $\mathbf{q}$ is the vector of heat flux, a general expression for which is given by Equation (2.3) in [2]. The remaining notation is the same as in [1].

Assuming from the nature of the problem that $T = T(z,t)$ and making the transition, as in [1], to nondimensional variables (by additional introduction of T^* as a scale for T), we will have instead of (1.1)

$$\frac{\partial T}{\partial t} = C^2 \frac{\partial^2 T}{\partial z^2} + F(z, t) \qquad (1.2)$$

where

$$F(z, t) = \frac{\gamma\varkappa}{1 + Zs} \left\{ \frac{1}{R^{(2)}} \left[\left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial u_y}{\partial z}\right)^2 \right] + j_x (E_{0x} + u_y) + j_y (E_{0y} - u_x) \right\} \qquad (1.3)$$

$$\gamma = \frac{U_0^2}{C^\circ T^*_p}, \qquad C^2 = \frac{\varkappa}{R^{(2)} P^{(2)} (1 + Zs)}, \qquad \varkappa = \frac{C_p^\circ}{C_v^\circ}, \qquad P^{(2)} = \frac{\eta^{(2)} C_p^\circ}{\lambda^T}$$

The coefficient of thermal conductivity λ^T which enters into Prandtl number $P^{(2)}$ coincides with the coefficient of heat conductivity for partially ionized gas in the absence of a magnetic field [2]. For sufficiently high degrees of ionization this coefficient is determined by electrons only (for $s \ll 1$ by neutral components). Thus, by virtue of the geometry of the problem the magnetic field does not affect the temperature field through the mechanism of thermal conductivity (div $q = \partial q_z / \partial z$, but the magnetic field has no effect on heath transfer in its direction). However this influence is realized through the action of internal heat sources (1.3) connected with viscous dissipation and Joule heating. The problem reduces to solution of inhomogeneous equation of thermal conductivity (1.2) for the following initial and boundary conditions:

$$T(z, 0) = T_0 = \text{const}, \qquad T(\pm 1, t) = 0 \qquad (t > 0) \tag{1.4}$$

It is not difficult to check that the inhomogeneous part of the equation is an even function of z .

Therefore

$$T(z, t) = \sum_{n=0}^{\infty} T_n(t) \cos \lambda_n z \qquad \left(\lambda_n = \frac{2n+1}{2}\pi \quad (n = 0, 1, 2, 3, \ldots)\right) \tag{1.5}$$

Introducing

$$F(z, t) = \sum_{n=0}^{\infty} F_n(t) \cos \lambda_n z \qquad \left(F_n(t) = \int_{-1}^{+1} F(z, t) \cos \lambda_n z \, dz\right) \tag{1.6}$$

and substituting (1.5) and (1.6) into (1.2), we obtain the following equation for determination of $T_n(t)$

$$\frac{dT_n}{dt} + C^2 \lambda_n^2 T_n = F_n \qquad \left(T_n(0) = T_0 \int_{-1}^{+1} \cos \lambda_n z dz = \frac{2T_0}{\lambda_n}(-1)^n\right) \tag{1.7}$$

The solution of Equation (1.7) will be

$$T_n(t) = \exp(-C^2 \lambda_n^2 t) \left[\int_0^t F_n(\tau) \exp(C^2 \lambda_n^2 \tau) \, d\tau + \frac{2T_0}{\lambda_n}(-1)^n\right] \tag{1.8}$$

From this the desired general solution takes the form

$$T(z, t) = \sum_{n=0}^{\infty} \left[\frac{2T_0}{\lambda_n}(-1)^n + \int_0^t F_n(\tau) \exp(c^2 \lambda_n^2 \tau) \, d\tau\right] \exp(-C^2 \lambda_n^2 t) \cos \lambda_n z \tag{1.9}$$

It remains to construct $F(z,t)$ with the aid of expressions obtained in [1] for complex velocity $v(z, t) = u_x - iu_y$ and for current $J = j_x - ij_y$ and to carry out the quadratures which enter into (1.9).

2. In the following we assume for simplicity that the differential pressure is applied only along the x-axis and that an external electrical field is absent, i.e. [1]

$$\psi_0 = P_x, \qquad P_y = \varphi_0 = E_{0x} - iE_{0y} \equiv 0 \tag{2.1}$$

Then, after extensive calculations including integration and summation of trigonometric series, the presentation of which does not appear possible here, the final solution of the problem is written in the form

$$T(z, t) = T_\eta(z) + T_\sigma(z) + T_t(z, t) \tag{2.2}$$

The first two terms give the steady temperature regime and take into account viscous friction through T_η and Joule heating through T_σ .

$$T_\eta = \frac{\gamma P^{(2)}(\Delta_1^2 + \Delta_2^2)(r_1^2 + r_2^2)}{8(\cosh^2 r_1 \cos^2 r_2 + \sinh^2 r_1 \sin^2 r_2)} \left(\frac{\cos 2r_2 - \cos 2r_2 z}{r_2^2} + \frac{\cosh 2r_1 - \cosh 2r_1 z}{r_1^2}\right) \tag{2.3}$$

$$T_\sigma = \frac{\Delta_1 \gamma P_x R^{(2)} p^{(2)}}{2}(1 - z^2) - \frac{\gamma R^{(2)} p^{(2)} (\Delta_1^2 + \Delta_2^2)}{\cosh^2 r_1 \cos^2 r_2 - \sinh^2 r_1 \sin^2 r_2} \times \tag{2.4}$$

$$\times \left[\frac{1}{8(m_1^2 + m_2^2)}\left(\frac{m_1}{R^{(2)}} + \frac{m_2}{R^{(4)}}\right)\left(\frac{\cos 2r_2 - \cos 2r_2 z}{r_2^2} - \frac{\cosh 2r_1 - \cosh 2r_1 z}{r_1^2}\right) + \right.$$

$$\left. + A^* (\cosh r_1 \cos r_2 - \cosh r_1 z \cos r_2 z) - A^{**} (\sinh r_1 \sin r_2 - \sinh r_1 z \sin r_2 z)\right]$$

Here

$$r_{1,2} = \left(\frac{\sqrt{m_1^2 + m_2^2} \pm m_1}{2(m_1^2 + m_2^2)}\right)^{1/2} \tag{2.5}$$

$$m_1 = \frac{1}{M^{(2)2}} - \omega_e \tau_0 \left[\frac{1}{M^{(4)2}} - 2(1-s)\,\omega_i \tau_{ia}\left(\frac{1}{M^{(2)2}} - \frac{1}{M_i^{(2)2}}\right)\right]$$

$$m_2 = \frac{1}{M^{(4)2}} + \omega_e \tau_0 \left[\frac{1}{M^{(2)2}} + 2(1-s)\,\omega_i \tau_{ia}\left(\frac{1}{M^{(4)2}} - \frac{1}{M_i^{(4)2}}\right)\right] \tag{2.6}$$

$$\Delta_1 = \frac{P_x}{N}\left[1 + \frac{2(1-s)^2}{1 + Zs}\,\omega_i \tau_{ia} \omega_e \tau_0\right], \qquad \Delta_2 = \frac{P_x \omega_e \tau_0}{N(1 + Zs)} \tag{2.7}$$

$$A^* = \frac{1}{R^{(4)}} \sinh r_1 \sin r_2 - \frac{1}{R^{(2)}} \cosh r_1 \cos r_2 + \frac{P_x}{\Delta_1^2 + \Delta_2^2}\left[\cosh r_1 \cos r_2 (m_2 \Delta_2 - m_1 \Delta_1) + \right.$$

$$\left. + \sinh r_1 \sin r_2 (m_2 \Delta_1 + m_1 \Delta_2)\right] \tag{2.8}$$

$$A^{**} = \frac{1}{R^{(4)}} \cosh r_1 \cos r_2 + \frac{1}{R^{(2)}} \sinh r_1 \sin r_2 + \frac{P_x}{\Delta_1^2 + \Delta_2^2}\left[\cosh r_1 \cos r_2 (m_2 \Delta_1 + \right.$$

$$\left. + m_1 \Delta_2) - \sinh r_1 \sin r_2 (m_2 \Delta_2 - m_1 \Delta_1)\right]$$

We note here that taking into account Larmor precession of charged particles leads to oscillatory character of steady temperature. If $\omega_e \tau_e \ll 1$, is introduced into the expression obtained, then the aperiodic solution results which corresponds to the temperature regime of the Hartmann problem when the electrical field is equal to zero

$$T_\eta + T_\sigma = \frac{\gamma R^{(0)2} p^{(0)} P_x^2}{M^{(0)2}}\left[\frac{1 - z^2}{2} \frac{2(\cosh M^{(0)} - \cosh M^{(0)} z)}{M^{(0)2} \cosh M^{(0)}} + \frac{\cosh 2M^{(0)} - \cosh 2M^{(0)} z}{4M^{(0)2} \cosh^2 M^{(0)}}\right. \tag{2.9}$$

The temperature pattern which corresponds to the other regime of the Hartmann problem was obtained in [3 to 5]. We will write the expression for the transition regime

$$T_l(z, t) = \sum_{n=0}^{\infty}\left[\frac{2T_\theta}{\lambda_n}(-1)^n - \int_{-1}^{+1}(T_\eta + T_\sigma)\cos\lambda_n z\, dz - \sum_{k=0}^{\infty} W_{kn}^{(1)} - \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} W_{kln}^{(3)}\right] \times$$

$$\times \exp(-C^2 \lambda_n^2 t)\cos\lambda_n z + \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{\infty}(W_{kn}^{(1)} \cos \Omega_k^{(2)} t + W_{kn}^{(2)} \sin \Omega_k^{(2)} t) \times \right.$$

$$\times \exp(-\Omega_k^{(1)} t) + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[W_{kln}^{(3)} \cos(\Omega_k^{(2)} - \Omega_l^{(2)})\, t + W_{kln}^{(4)} \sin(\Omega_k^{(2)} - \Omega_l^{(2)})\, t\right] \times$$

$$\left. \times \exp\left[-(\Omega_k^{(1)} + \Omega_l^{(1)})\, t\right]\right\} \cos\lambda_n z \tag{2.10}$$

The constants $W_{kn}^{(1)}$, $W_{kn}^{(2)}$, $W_{kln}^{(3)}$ and $W_{kln}^{(4)}$ have the form

$$W_{kn}^{(1)} + iW_{kn}^{(2)} = \frac{(-1)^{n+1}\gamma\varkappa}{(1+Zs)\,(C^2\lambda_n^2 - \Omega_k^{(1)} - i\Omega_k^{(2)})}\Big\{\Big[P_x\,(d_k^{(1)} - id_k^{(2)}) +$$

$$+ (\beta_k^{(1)} - i\beta_k^{(2)})\,(\Delta_1 - i\Delta_2)\,\varepsilon^{kn} - \frac{\lambda_n\,(\Delta_1 - i\Delta_2)\,(\sinh r_1 \cos r_2 + i\cosh r_1 \sin r_2)}{\cosh r_1 \cos r_2 + i \sinh r_1 \sin r_2} \times$$

$$\times \Big\langle\Big[(\beta_k^{(1)} - i\beta_k^{(2)}) + \frac{d_k^{(1)} - id_k^{(2)}}{m_1 - im_2}\left(\frac{1}{R^{(2)}} - i\frac{1}{R^{(4)}}\right)\Big][2r_1\,(I_{kn}^{(1)} + I_{kn}^{(2)}) -$$

$$- i\,(I_{kn}^{(3)} - I_{kn}^{(4)})] + \frac{2\lambda_k}{R^{(2)}}(r_2 - ir_1)(d_k^{(1)} - id_k^{(2)})[2r_1\,(I_{kn}^{(1)} - I_{kn}^{(2)}) - i\,(I_{kn}^{(3)} + I_{kn}^{(4)})]\Big\rangle\Big\} \quad (2.11)$$

$$W_{kln}^{(3)} + iW_{kln}^{(4)} = \frac{2\,(-1)^{n+1}\gamma\varkappa\lambda_k\lambda_l\lambda_n}{(1+Zs)[C^2\lambda_n^2 - \Omega_k^{(1)} - \Omega_l^{(1)} - i\,(\Omega_k^{(2)} - \Omega_l^{(2)})]} \times$$

$$\times [\beta_k^{(1)}d_l^{(1)} + \beta_k^{(2)}d_l^{(2)} + \frac{1}{R^{(2)}}(d_k^{(1)}d_l^{(1)} + d_k^{(2)}d_l^{(2)})\,(\lambda_k^2 + \lambda_l^2 - \lambda_n^2)] \times$$

$$\times \frac{1}{(\lambda_k^2 + \lambda_l^2 - \lambda_n^2))^2 - 4\lambda_k^2\lambda_l^2} \quad (2.12)$$

$$\beta_k^{(1)} - i\beta_k^{(2)} = \frac{2\,(\Delta_1 + i\Delta_2)\,N^*}{\lambda_k\,(\lambda_k^2/M_\delta^{*2} + 1)}\left[1 + \lambda_k^2\left(\frac{1}{R_\delta^*} - \frac{1}{R^*}\right)\right]$$

$$d_k^{(1)} - id_k^{(2)} = \frac{2\,(\Delta_1 + i\Delta_2)}{\lambda_k\,(\lambda_k^2/M_\delta^{*2} + 1)}, \quad \varepsilon^{kn} = \begin{cases} 1 & \text{for } k = n \\ 0 & \text{for } k \neq n \end{cases} \quad (i = \sqrt{-1}) \quad (2.13)$$

$$I_{kn}^{(1,2)} = \frac{\lambda_k \mp r_2}{[r_1^2 + (\lambda_k \mp r_2)^2 + \lambda_n^2]^2 - 4\,(\lambda_k \mp r_2)^2\,\lambda_n^2}$$

$$I_{kn}^{(3,4)} = \frac{r_1^2 - (\lambda_k \mp r_2)^2 + \lambda_n^2}{[r_1^2 + (\lambda_k \mp r_2)^2 + \lambda_n^2]^2 - 4\,(\lambda_k \mp r_2)^2\,\lambda_n^2} \quad (2.14)$$

$$\Omega_k^{(1)} = \frac{N\,\{(1 + m_1\lambda_k^2)\,[1 + 2\,(1-s)^2\,\omega_i\tau_{ia}\omega_e\tau_0] + m_2\omega_e\tau_0\lambda_k^2\}}{[1 + 2\,(1-s)^2\,\omega_i\tau_{ia}\omega_e\tau_0]^2 + (\omega_e\tau_0)^2}$$

$$\Omega_k^{(2)} = \frac{N\,\{m_2\lambda_k^2\,[1 + 2\,(1-s)^2\,\omega_i\tau_{ia}\omega_e\tau_0] - \omega_e\tau_0\,(1 + m_1\lambda_k^2)\}}{[1 + 2\,(1-s)^2\,\omega_i\tau_{ia}\omega_e\tau_0]^2 + (\omega_e\tau_0)^2} \quad (2.15)$$

The obtained expression is very cumbersome and it includes both the aperiodic and the periodic parts with respect to time. The appearance of the latter is due to Larmor precession of charged particles. In fact, assuming $\omega_e\tau_0 \ll 1$, we have for cyclic frequencies

$$\Omega_k^{(2)} = 0, \qquad \Omega_k^{(2)} - \Omega_l^{(2)} = 0 \quad (2.16)$$

Assuming on the other hand $\omega_i\tau_i\theta \ll 1$ $(\omega_i\tau_{ia} \ll 1)$, we obtain

$$\Omega_k^{(2)} = -N\frac{\omega_e\tau_0}{1 + (\omega_e\tau_0)^2}, \qquad \Omega_k^{(2)} - \Omega_l^{(2)} = 0 \quad (2.17)$$

Thus, Larmor precession of ions leads to periodicity of the last sum in (2.10) and also to a spectrum of frequencies of oscillation in contrast to the case $\omega_i\tau_i\theta \ll 1$. It is also interesting to observe the effect of magnetic field on the decay coefficient of the periodic part of the transition regime $\Omega_k^{(1)}$ and $\Omega_k^{(1)} + \Omega_l^{(1)}$ (the coefficient of decay of the aperiodic part $C^2\lambda_n^2$ does not depend on the magnetic field). Assuming the value $B_0 = 0$ in (2.15), we obtain for purely hydrodynamic flow

$$\Omega_k^{(1)} = \lambda_k^2 / R^{(0)} \qquad (2.18)$$

For the isotropic magnetohydrodynamic case $(\omega_e\tau_0 \ll 1)$

$$\Omega_k^{(1)} = N + \lambda_k^2 / R^{(0)} \qquad (2.19)$$

i.e. an increase in the decay coefficient is observed.

Consideration of the effect of anisotropy in conductivity ($\omega_e\tau_0$ is finite, $\omega_i\tau_i\theta \ll 1$) gives

$$\Omega_k^{(1)} = \frac{N}{1 + (\omega_e\tau_0)^2} + \frac{\lambda_k^2}{R^{(0)}} \qquad (2.20)$$

i.e. for equal parameters of magnetic interacion N we have a decrease in $\Omega_k^{(1)}$ in comparison with the isotropic case (generally, N also increases with increase in the magnetic field, as B_0^2). Finally, consideration of Larmor precession of ions ($\omega_i\tau_i\theta$ and $\omega_i\tau_{ia}$ are finite quantities) gives

$$\Omega_k^{(1)} = \frac{N}{1 + (\omega_e\tau_{ei})^2} + \frac{\lambda_k^2}{R^{(0)}\,[1 + {}^4/_9(\omega_i\tau_i)^2]} \qquad (2.21)$$

for completely ionized medium $(s = 1)$ and

$$\Omega_k^{(1)} = \frac{N\,(1 + 2\omega_i\tau_{ia}\omega_e\tau_{ea})}{(1 + 2\omega_i\tau_{ia}\omega_e\tau_{ea})^2 + (\omega_e\tau_{ea})^2} + \frac{\lambda_k^2}{R^{(0)}} \qquad (2.22)$$

for weakly ionized medium $(s \ll 1)$. Comparing (2.21) and (2.20) we see that for completely ionized medium consideration of Larmor precession of ions lowers the coefficient of decay. Finally, comparing (2.22) and (2.20) and taking into consideration that

$$\omega_i\tau_{ia} \sim \left(\frac{m_e}{m_i}\right)^{1/2}\omega_e\tau_{ea} \qquad \left(\frac{m_e}{m_i}\right)^{1/2} \ll 1 \qquad \left(\begin{matrix} m_e \text{ mass of electron} \\ m_i \text{ mass of ion}\end{matrix}\right)$$

we have the analogous picture for the weakly ionized medium also.

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